Birzeit University

Mathematics Department

Order of convergence | Math330 | 2018/2019

(Q) Let p be a fixed point of $g(x)$.

(a) show that if $g'(p) = g''(p) = \cdots = g^{(k-1)}(p) = 0$ and $g^{(k)}(p) \neq 0$, then the fixed point iteration of $g(x)$ will converge to p with $R = k$ and $A = |\frac{g^{(k)}(p)}{k!}|$

(b) Use part (a) to show that if p is a simple root of $f(x)$, then Newton's iteration will converge to p with $R = 2$ and $A = |\frac{f''(p)}{2f'(p)}|$

Proof: (a) Apply Taylor's expansion of $g(x)$ about $x = p$

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \cdots + \frac{g^{(k-1)}(p)}{(k-1)!}(x-p)^{k-1} + \frac{g^{(k)}(c)}{k!}(x-p)^k$$

but $g(p) = p$ and $g'(p) = g''(p) = \cdots = g^{(k-1)}(p) = 0$

$\rightarrow \; g(x) = p + \frac{g^{(k)}(c)}{k!}(x-p)^k$

Substitute $x = p_n \rightarrow \; g(p_n) = p + \frac{g^{(k)}(c)}{k!}(p_n - p)^k$, c between p_n and p

but $g(p_n) = p_{n+1} \; \rightarrow p_{n+1} - p = \frac{g^{(k)}(c)}{k!}(p_n - p)^k \; \rightarrow \frac{p_{n+1}-p}{(p_n-p)^k} = \frac{g^{(k)}(c)}{k!} \; \rightarrow \frac{|E_{n+1}|}{|E_n|^k} = |\frac{g^{(k)}(c)}{k!}|$

now take limit an $n \rightarrow \infty$ and considering that $c \approx p$ when $n \rightarrow \infty$

$\rightarrow \; \lim_{n\rightarrow\infty} \frac{|E_{n+1}|}{|E_n|^k} = |\frac{g^{(k)}(p)}{k!}| \; \rightarrow R = k$ and $A = |\frac{g^{(k)}(p)}{k!}|$

(b) We know that Newton's iteration is a special case of FPI with $g(x) = x - \frac{f(x)}{f'(x)}$

Therefore, based on part (a), we only need to prove that $g'(p) = 0$ but $g''(p) \neq 0$

Recall that p is a simple root of $f(x)$ mean $f(p) = 0$ and $f'(p) \neq 0$

now, $g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2} \; \rightarrow g'(p) = 0$

now, $g''(x) = \frac{(f'(x))^2 \; [f(x)f'''(x) + f''(x)f'(x)]}{(f'(x))^4} = \frac{f(x)f'''(x) + f''(x)f'(x)}{(f'(x))^2} \; \rightarrow g''(p) = \frac{f''(p)}{f'(p)} \neq 0$

$\rightarrow \; R = 2$ and $A = |\frac{g''(p)}{2!}| = |\frac{f''(p)}{2f'(p)}|$